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A PROPERTY OF FINITE p-GROUPS WITH TRIVIAL MULTIPLICATOR

BY

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ABSTRACT. A sufficient condition for a finite 2-generator p-group to have nontrivial multiplicator is given. To show that this result is best possible, a finite 2-group with trivial multiplicator is constructed.

1. Introduction. In recent years the interest in finite p-groups with trivial multiplicator has increased considerably. This is mainly due to the work of D. L. Johnson and J. W. Wamsley.

It is well known (see for example [4]) that finite p-groups needing at least four generators have nontrivial multiplicators (a result recently generalised by Gaschütz and Newman [3]) and D. L. Johnson has recently shown that groups of prime exponent have nontrivial multiplicator (provided they are noncyclic, of course). In [5] we gave a sufficient condition for a finite three-generator p-group to have nontrivial multiplicator and, since the finite metacyclic p-groups with trivial multiplicator have been classified by Wamsley [6], we aim in the present note to extend the result of [5] and prove

THEOREM A. *Let G be a finite nonmetacyclic p-group with two generators.*

(i) *If p is odd and the derived group of G needs at most two generators then G has nontrivial multiplicator.*

(ii) *If* $p = 2$ *and the derived group of G is cyclic, G has nontrivial multiplicator.*

(iii) *If* $p = 2$, *if the derived group of G is two-generator and if neither of the invariants of the derived factor-group of G is* 1, *G has nontrivial multiplicator.*
□

Notation 1.1. For elements x and y of some group we represent the elements $y^{-1}xy$ and $x^{-1}y^{-1}xy$ by x^y and $[x, y]$ respectively. If G is a group and U and V are subgroups of G then $[U, V]$ denotes the subgroup of G generated by all ele-

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ments $[u, v]$ with u in U and v in V; in particular, the derived group of G is $[G, G]$ and is denoted by G'.

Finally, the lower central series of a group G is denoted by $G = G_1 \geqslant G' = G_2 \geqslant G_3 \geqslant \cdots$. All other notation, where not explained, will be standard. □

2. **Preliminaries.** There are many characterisations of the multiplicator, $M(G)$, of a finite group G but the two that have proved most useful in this line of work are as follows:

2.1. *Let G be a finite group and $G = F/R$ a presentation for G as a factor-group of the free group F. Then $M(G) \cong (F' \cap R)/[F, R]$.* □

2.2. *Let G be a finite group. A group H is said to be a representing group of G if it contains a subgroup L in $H' \cap Z(H)$ such that $H/L \cong G$ and $L \cong M(G)$.* □

We require a series of lemmas, all of which are essential to our method of approach. The first of these is due to J. W. Wamsley (as yet unpublished).

LEMMA 2.3. *Let G be a finite group with trivial multiplicator. Then, for all $j \geqslant 2$, $M(G/G_j) \cong G_j/G_{j+1}$.*

PROOF. Consider the group $H = G/G_{j+1}$. Then if $K = G_j/G_{j+1}$, $K \leqslant H' \cap Z(H)$ so that K is a subgroup of $M(H/K) = M(G/G_j)$ by a well-known result of Schur (see [4]).

If $G = F/R$ is a presentation for G with F free it follows from 2.1 that $F' \cap R = [F, R]$, so that the proof is completed by observing that the map θ of $F_jR/F_{j+1}R$ into $(F' \cap R)F_j/(F' \cap R)F_{j+1}$ defined by $(fF_{j+1}R)\theta = f(F' \cap R)F_{j+1}$ for f in F_j is an epimorphism. □

LEMMA 2.4 (JONES [5]). *Let G be a finite group and K any normal subgroup of G. Set $H = G/K$. Then $d(M(H))$ is no more than $d(M(G)) + d(G' \cap K)$.* □

The next two results are due to Blackburn [1] and so their proofs are omitted here.

LEMMA 2.5. *Let G be a finite nonmetacyclic p-group with two generators. Then $H = G/\Phi(G')G_3$ has defining relations*

$$[a, b] = c, \quad a^{p^\lambda} = b^{p^\mu} = 1, \quad [a, c] = [b, c] = 1, \quad c^p = 1$$

in terms of generators a and b, where λ and μ are the invariants of G/G'. □

COROLLARY 2.6. *Let G be a finite nonmetacyclic p-group with two generators.*

(i) *If p is odd, the group X defined by*

$$X = \langle a, b \colon a^p = b^p = [a, b, a] = [a, b, b] = 1 \rangle$$

is an epimorphic image of G.

(ii) *If $p = 2$, the group Y defined by*

$$Y = \langle a, b \colon [a, b] = c, a^2 = b^4 = c^2 = 1 = [a, c] = [b, c] \rangle$$

is an epimorphic image of G. □

The final pair of lemmas require rather technical proofs so that, for the sake of brevity, we only sketch their proofs.

LEMMA 2.7. (i) *The group X of Corollary 2.6 has multiplicator $Z_p \times Z_p$.*
(ii) *The group Y of Corollary 2.6 has multiplicator $Z_2 \times Z_2$.*

PROOF. (i) It follows from the well-known result of Gaschütz, Neubüser and Yen [2] that $|M(X)| \leqslant p^2$. Consider the group H defined by

$$H = \langle x, y \colon [x, y] = d, [d, x] = e, [d, y] = f, \\ x^p = y^p = d^p = e^p = f^p = 1, \\ [e, x] = [e, y] = [f, x] = [f, y] = [e, f] = 1 \rangle$$

and the subgroup $L = \langle e, f \rangle$. Then it is easy to see that $H/L \cong X$ so that L is a subgroup of $M(X)$ (see [4]).

Since $L \cong Z_p \times Z_p$, the result now follows.

(ii) This follows as for (i). □

LEMMA 2.8. *Let G and H be as in Lemma 2.5. Then provided neither λ nor μ is 1, $M(H)$ is a three-generator group.*

PROOF. This follows as for Lemma 2.7. □

3. **Proof of Theorem A.** We prove Theorem A in three steps as follows:

I. Suppose firstly that G' is cyclic. Then by Lemmas 2.4 and 2.7 and Corollary 2.6 we have

$$2 = d(M(G/N)) \leqslant d(M(G)) + d(G' \cap N) \leqslant d(M(G)) + 1,$$

for some N normal in G.

Hence $d(M(G)) \geqslant 1$ and $M(G)$ is nontrivial.

II. Now suppose $d(G') = 2$ and suppose that neither of the invariants of G/G' is 1. The work of Blackburn (see for example [4, III, 7.9]) shows that G is metabelian and by Lemmas 2.4, 2.5 and 2.8 we have

$$3 = d(M(G/N)) \leqslant d(M(G)) + d(G' \cap N) \leqslant d(M(G)) + 2,$$

for some N normal in G.

Hence $d(M(G)) \geqslant 1$ and $M(G)$ is nontrivial again.

III. Finally suppose $d(G') = 2$, p is odd and $G/G' \cong Z_p \times Z_{p^\alpha}$ for some α. As above, G is metabelian. Assume $G = \langle a, b \rangle$ where $a^p, b^{p^\alpha} \in G'$. If we assume G has trivial multiplicator it follows that, since $M(G/G') \cong Z_p$, $G'/G_3 \cong Z_p$ (see Lemma 2.3) so that G_j/G_{j+1} has exponent p for all $j \geqslant 2$.

Now if $a^p \in G' \backslash G_3$, it follows by induction on j that $G_j = \langle a^{p^{j-1}}, G_{j+1} \rangle$ for all $j \geqslant 2$ so that G' is cyclic and we have a contradiction. Similarly if $b^{p^\alpha} \in G' \backslash G_3$. Hence $a^p, b^{p^\alpha} \in G_3$.

Consider G/G_3. This has presentation

$$\langle x, y \colon z = [x, y], x^p = y^{p^\alpha} = z^p = 1, [x, z] = [y, z] = 1 \rangle$$

and it follows as in the proof of Lemma 2.7 that this group has multiplicator $Z_p \times Z_p$. Hence Lemma 2.3 shows that $G_3/G_4 \cong Z_p \times Z_p$. We may therefore assume that $G' = \langle c, d \rangle$, where $c = [a, b]$ and $d \in G_3 \backslash G_4$.

Clearly $G_3 = \langle c^p, d \rangle$ so that $c^p \notin G_4$. Now $[a^p, b] \equiv 1 \pmod{G_4}$. But since G is metabelian,

$$[a^p, b] \equiv [a, b]^p [a, b, a]^{\frac{1}{2}p(p-1)} \pmod{G_4}.$$

so that $[a^p, b] \equiv [a, b]^p \pmod{G_4}$. Hence $c^p \in G_4$ and we have a contradiction. The proof of Theorem A is now complete.

4. An example. Theorem A together with the result of [5] shows us that all nonmetacyclic groups with trivial multiplicator have derived group needing at least three generators, with the possible exceptions of 2-groups with one invariant of their derived factor-group being 1. There are plenty of examples of p-groups with trivial multiplicator and 3-generator derived group, but none are known which have trivial multiplicator and derived group needing more than three generators. It seems reasonable, therefore, to ask whether such groups exist.

Finally, we construct a 2-group with trivial multiplicator whose derived factor-group is $Z_2 \times Z_4$.

Let $A = \langle g \rangle \times \langle h \rangle$ be the direct product of two cyclic groups of order 4 and form B, the semidirect product of a cyclic group, $\langle a \rangle$, of order 4 and A under the action $g^a = g^{-1}h^2$, $h^a = h^{-1}$, amalgamating $\langle a^2 \rangle$ and $\langle g^2 \rangle$.

Then $B = \langle a, g, h\rangle$ and has defining relations

$$a^4 = g^4 = h^4 = 1, \quad [g, h] = 1,$$
$$[g, a] = g^2h^2, \quad [h, a] = h^2, \quad a^2 = g^2.$$

Finally, the group G we require is formed by the semidirect product of $\langle b\rangle$, a cyclic group of order 16, and B under the action $a^b = ag$, $g^b = gh$, $h^b = h$, amalgamating $\langle b^4\rangle$ and $\langle h\rangle$.

Then $G = \langle a, b, g, h\rangle$ and has defining relations

$$a^4 = b^{16} = g^4 = h^4 = 1, \quad [g, h] = 1,$$
$$[g, a] = g^2h^2, \quad [h, a] = h^2,$$
$$[a, b] = g, \quad [g, b] = h, \quad [h, b] = 1,$$
$$a^2 = g^2, \quad b^4 = h.$$

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